13-5 Law of Cosines

Main Ideas

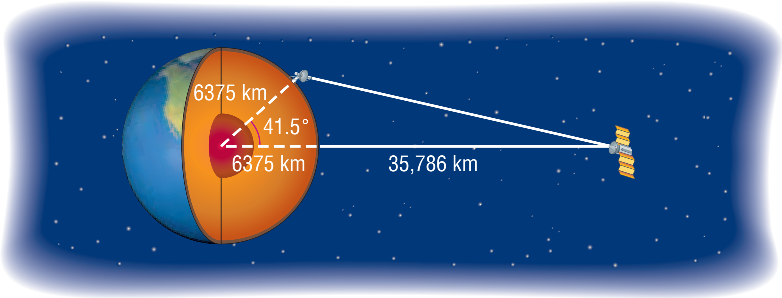
- Solve problems by using the Law of Cosines.
- Determine whether a triangle can be solved by first using the Law of Sines or the Law of Cosines.

New Vocabulary

Law of Cosines

GET READY for the Lesson

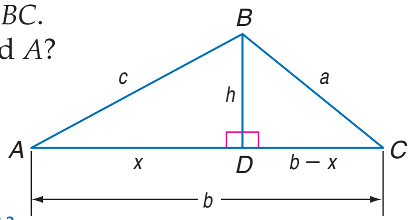
A satellite in a *geosynchronous orbit* about Earth appears to remain stationary over one point on the equator. A receiving dish for the satellite can be directed at one spot in the sky. The satellite orbits 35,786 kilometers above the equator at 87°W longitude. The city of Valparaiso, Indiana, is located at approximately 87°W longitude and 41.5°N latitude.



If the radius of Earth is about 6375 kilometers, you can use trigonometry to determine the angle at which to direct the receiver.

Law of Cosines Problems such as this, in which you know the measures of two sides and the included angle of a triangle, cannot be solved using the Law of Sines. You can solve problems such as this by using the **Law of Cosines**.

To derive the Law of Cosines, consider $\triangle ABC$. What relationship exists between a, b, c, and A?



$a^2 = (b - x)^2 + h^2$	Use the Pythagorean Theorem for $\triangle DBC$.
$= b^2 - 2bx + x^2 + h^2$	Expand $(b - x)^2$.
$= b^2 - 2bx + c^2$	In $\triangle ADB$, $c^2 = x^2 + h^2$.
$= b^2 - 2b(c \cos A) + c^2$	$\cos A = \frac{x}{c}$, so $x = c \cos A$.
$= b^2 + c^2 - 2bc \cos A$	Commutative Property

Study Tip

You can apply the Law of Cosines to a triangle if you know the measures of two sides and the included angle, or the measures of three sides.

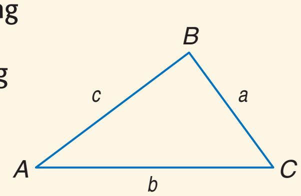
KEY CONCEPT — *Law of Cosines*

Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of sides, and opposite angles with measures A, B, and C, respectively. Then the following equations are true.

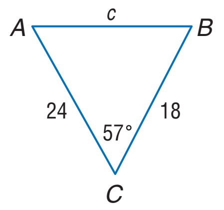
$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$



EXAMPLE 1 Solve a Triangle Given Two Sides and Included Angle



Solve $\triangle ABC$.

Begin by using the Law of Cosines to determine c.

$c^2 = a^2 + b^2 - 2ab \cos C$ — Law of Cosines

$c^2 = 18^2 + 24^2 - 2(18)(24) \cos 57°$ — $a = 18$, $b = 24$, and $C = 57°$

$c^2 \approx 429.4$ — Simplify using a calculator.

$c \approx 20.7$ — Take the square root of each side.

Next, you can use the Law of Sines to find the measure of angle A.

$\frac{\sin A}{a} = \frac{\sin C}{c}$ — Law of Sines

$\frac{\sin A}{18} \approx \frac{\sin 57°}{20.7}$ — $a = 18$, $C = 57°$, and $c < 20.7$

$\sin A \approx \frac{18 \sin 57°}{20.7}$ — Multiply each side by 18.

$\sin A \approx 0.7293$ — Use a calculator.

$A \approx 47°$ — Use the $\sin^{-1}$ function.

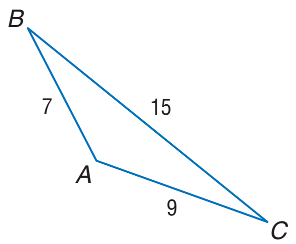
The measure of angle B is approximately $180° - (57° + 47°)$ or $76°$. Therefore, $c \approx 20.7$, $A \approx 47°$, and $B \approx 76°$.

> **Study Tip**
>
> **Alternative Method**
>
> After finding the measure of c in Example 1, the Law of Cosines could be used again to find a second angle.

CHECK Your Progress

1. Solve $\triangle FGH$ if $m\angle G = 82°$, $f = 6$, and $h = 4$.

EXAMPLE 2 Solve a Triangle Given Three Sides



Solve $\triangle ABC$.

Use the Law of Cosines to find the measure of the largest angle first, angle A.

$a^2 = b^2 + c^2 - 2bc \cos A$ — Law of Cosines

$15^2 = 9^2 + 7^2 - 2(9)(7) \cos A$ — $a = 15$, $b = 9$, and $c = 7$

$15^2 - 9^2 - 7^2 = -2(9)(7) \cos A$ — Subtract 9^2 and 7^2 from each side.

$\frac{5^2 - 9^2 - 7^2}{-2(9)(7)} = \cos A$ — Divide each side by $-2(9)(7)$.

$-0.7540 \approx \cos A$ — Use a calculator.

$139° \approx A$ — Use the $\cos^{-1}$ function.

You can use the Law of Sines to find the measure of angle B.

$\frac{\sin B}{b} = \frac{\sin A}{a}$ — Law of Sines

$\frac{\sin B}{9} \approx \frac{\sin 139°}{15}$ — $b = 9$, $A \approx 139°$, and $a = 15$

$\sin B \approx \frac{9 \sin 139°}{15}$ — Multiply each side by 9.

$\sin B \approx 0.3936$ — Use a calculator.

$B \approx 23°$ — Use the $\sin^{-1}$ function.

The measure of angle C is approximately $180° - (139° + 23°)$ or $18°$. Therefore, $A \approx 139°$, $B \approx 23°$, and $C \approx 18°$.

Study Tip

Sides and Angles

When solving triangles, remember that the angle with the greatest measure is always opposite the longest side. The angle with the least measure is always opposite the shortest side.

CHECK Your Progress

2. Solve $\triangle FGH$ if $f = 2$, $g = 11$, and $h = 1$.

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Choose the Method To solve a triangle that is *oblique*, or having no right angle, you need to know the measure of at least one side and any two other parts. If the triangle has a solution, then you must decide whether to begin solving by using the Law of Sines or the Law of Cosines. Use the chart to help you choose.

CONCEPT SUMMARY *Solving an Oblique Triangle*

Given	Begin by Using
two angles and any side	Law of Sines
two sides and an angle opposite one of them	Law of Sines
two sides and their included angle	Law of Cosines
three sides	Law of Cosines

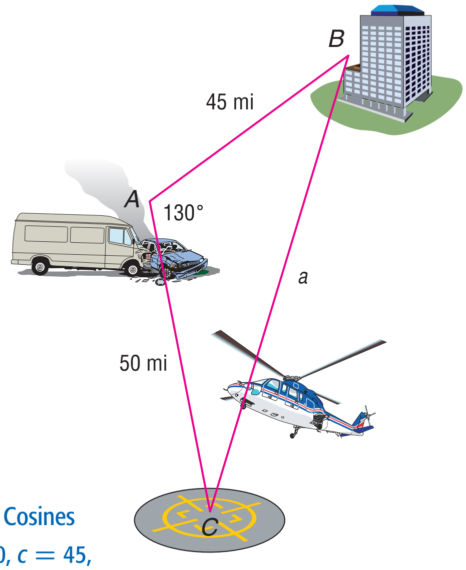
Real-World Link

Medical evacuation (Medevac) helicopters provide quick transportation from areas that are difficult to reach by any other means. These helicopters can cover long distances and are primary emergency vehicles in locations where there are few hospitals.

Source: The Helicopter Education Center

Real-World EXAMPLE Apply the Law of Cosines

3 **EMERGENCY MEDICINE A medical rescue helicopter has flown from its home base at point C to pick up an accident victim at point A and then from there to the hospital at point B. The pilot needs to know how far he is now from his home base so he can decide whether to refuel before returning. How far is the hospital from the helicopter's base?**



You are given the measures of two sides and their included angle, so use the Law of Cosines to find a.

$a^2 = b^2 + c^2 - 2bc \cos A$ — Law of Cosines

$a^2 = 50^2 + 45^2 - 2(50)(45) \cos 130°$ — $b = 50$, $c = 45$, and $A = 130°$.

$a^2 \approx 7417.5$ — Use a calculator to simplify.

$a \approx 86.1$ — Take the square root of each side.

The distance between the hospital and the helicopter base is approximately 86.1 miles.

CHECK Your Progress

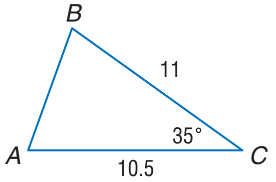
3. As part of training to run a marathon, Amelia ran 6 miles in one direction. She then turned and ran another 9 miles. The two legs of her run formed an angle of 79°. How far was Amelia from her starting point at the end of the 9-mile leg of her run?

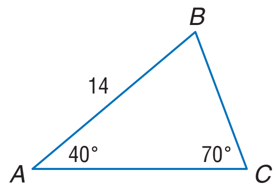
Math Online Extra Examples at algebra2.com

✓ CHECK Your Understanding

Examples 1, 2
(pp. 794–795)

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

1. 

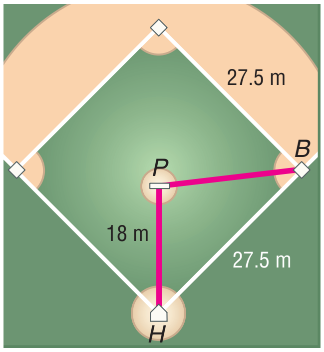
2. 

3. $A = 42°, b = 57, a = 63$

4. $a = 5, b = 12, c = 13$

Example 3
(p. 795)

BASEBALL **For Exercises 5 and 6, use the following information.**
In Australian baseball, the bases lie at the vertices of a square 27.5 meters on a side and the pitcher's mound is 18 meters from home plate.



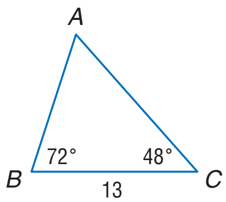
5. Find the distance from the pitcher's mound to first base.

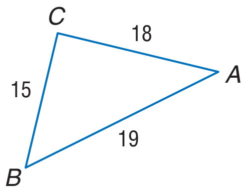
6. Find the angle between home plate, the pitcher's mound, and first base.

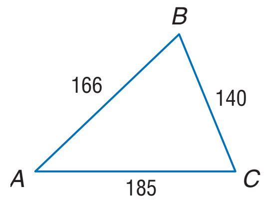
Exercises

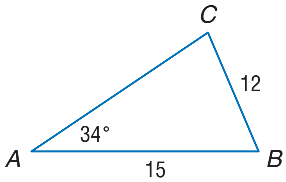
HOMEWORK HELP	
For Exercises	**See Examples**
7–18	1, 2
19, 20	3

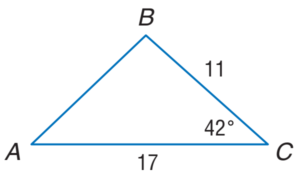
Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

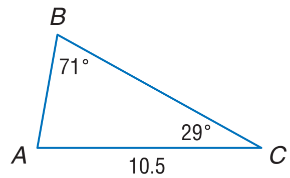
7. 

8. 

9. 

10. 

11. 

12. 

13. $a = 20, c = 24, B = 47°$

14. $a = 345, b = 648, c = 442$

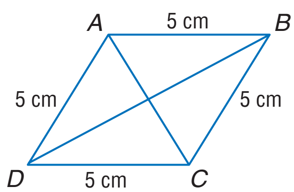
15. $A = 36°, a = 10, b = 19$

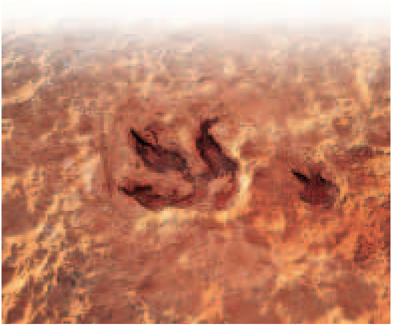
16. $A = 25°, B = 78°, a = 13.7$

17. $a = 21.5, b = 16.7, c = 10.3$

18. $a = 16, b = 24, c = 41$

19. **GEOMETRY** In rhombus $ABCD$, the measure of $\angle ADC$ is 52°. Find the measures of diagonals $\overline{AC}$ and $\overline{BD}$ to the nearest tenth.





Real-World Link

At digs such as the one at the Glen Rose formation in Texas, anthropologists study the footprints made by dinosaurs millions of years ago. *Locomoter* parameters, such as pace and stride, taken from these prints can be used to describe how a dinosaur once moved.

Source: Mid-America Paleontology Society

20. SURVEYING Two sides of a triangular plot of land have lengths of 425 feet and 550 feet. The measure of the angle between those sides is 44.5°. Find the perimeter and area of the plot.

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

21. $a = 8, b = 24, c = 18$

22. $B = 19°, a = 51, c = 61$

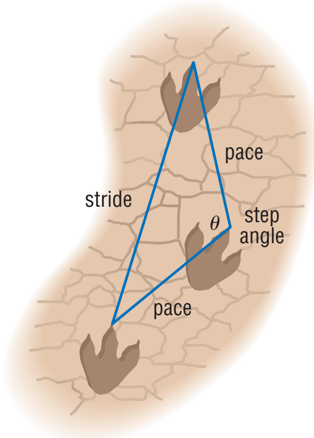
23. $A = 56°, B = 22°, a = 12.2$

24. $a = 4, b = 8, c = 5$

25. $a = 21.5, b = 13, C = 38°$

26. $A = 40°, b = 7, a = 6$

DINOSAURS **For Exercises 27–29, use the diagram at the right.**



27. An anthropologist examining the footprints made by a bipedal (two-footed) dinosaur finds that the dinosaur's average pace was about 1.60 meters and average stride was about 3.15 meters. Find the step angle θ for this dinosaur.

28. Find the step angle θ made by the hindfeet of a herbivorous dinosaur whose pace averages 1.78 meters and stride averages 2.73 meters.

29. An efficient walker has a step angle that approaches 180°, meaning that the animal minimizes "zig-zag" motion while maximizing forward motion. What can you tell about the motion of each dinosaur from its step angle?

EXTRA PRACTICE

See pages 921, 938.

Math Online

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30. AVIATION A pilot typically flies a route from Bloomington to Rockford, covering a distance of 117 miles. In order to avoid a storm, the pilot first flies from Bloomington to Peoria, a distance of 42 miles, then turns the plane and flies 108 miles on to Rockford. Through what angle did the pilot turn the plane over Peoria?

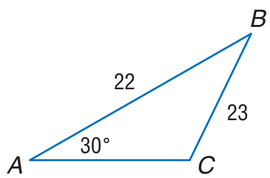
H.O.T. Problems

31. REASONING Explain how to solve a triangle by using the Law of Cosines if the lengths of

a. three sides are known.

b. two sides and the measure of the angle between them are known.

32. FIND THE ERROR Mateo and Amy are deciding which method, the Law of Sines or the Law of Cosines, should be used first to solve $\triangle ABC$.



Mateo	Amy
Begin by using the Law of Sines, since you are given two sides and an angle opposite one of them.	Begin by using the Law of Cosines, since you are given two sides and their included angle.

Who is correct? Explain your reasoning.

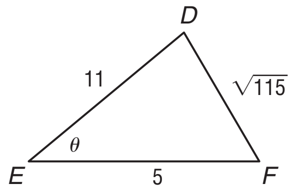
33. **OPEN ENDED** Give an example of a triangle that can be solved by first using the Law of Cosines.

34. **CHALLENGE** Explain how the Pythagorean Theorem is a special case of the Law of Cosines.

35. ***Writing in Math*** Use the information on page 793 to explain how you can determine the angle at which to install a satellite dish. Include an explanation of how, given the latitude of a point on Earth's surface, you can determine the angle at which to install a satellite dish at the same longitude.

STANDARDIZED TEST PRACTICE

36. **ACT/SAT** In $\triangle DEF$, what is the value of θ to the nearest degree?



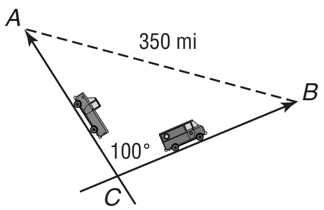
A 26°

B 74°

C 80°

D 141°

37. **REVIEW** Two trucks, A and B, start from the intersection C of two straight roads at the same time. Truck A is traveling twice as fast as truck B and after 4 hours, the two trucks are 350 miles apart. Find the approximate speed of truck B in miles per hour.



F 35 G 37 H 57 J 73

Spiral Review

38. **SANDBOX** Mr. Blackwell is building a triangular sandbox. He is to join a 3-meter beam to a 4 meter beam so the angle opposite the 4-meter beam measures 80°. To what length should Mr. Blackwell cut the third beam in order to form the triangular sandbox? Round to the nearest tenth. (Lesson 13-4)

Find the exact values of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point. (Lesson 13-3)

39. $(5, 12)$ 40. $(4, 7)$ 41. $(\sqrt{10}, \sqrt{6})$

Solve each equation or inequality. (Lesson 9-5)

42. $e^x + 5 = 9$ 43. $4e^x - 3 > -1$ 44. $\ln(x + 3) = 2$

GET READY for the Next Lesson

PREREQUISITE SKILL **Find one angle with positive measure and one angle with negative measure coterminal with each angle.** (Lesson 13-2)

45. 45° 46. 30° 47. 180°

48. $\frac{\pi}{2}$ 49. $\frac{7\pi}{6}$ 50. $\frac{4\pi}{3}$